Eigenspaces of Tournament Matrices
Overview

I. Preliminaries

II. Basic Tournament Properties

III. The Brualdi-Li Perron Eigenspace

IV. Purely Imaginary Eigenvalues

V. Open Questions
Definitions

- $A \in \mathcal{M}_n(\mathbb{R})$ is a **tournament matrix** if it satisfies
  \[ A + A^t = J_n - I_n, \quad A \geq 0, \quad A \circ A = A. \]

- $A \in \mathcal{M}_n(\mathbb{R})$ is a **generalized tournament matrix** if it satisfies
  \[ A + A^t = J_n - I_n, \quad A \geq 0. \]

- $A \in \mathcal{M}_n(\mathbb{R})$ is a **1-hypertournament matrix** if it satisfies
  \[ A + A^t = J_n - I_n. \]

- $A \in \mathcal{M}_n(\mathbb{R})$ is a **h-hypertournament matrix** if it satisfies
  \[ A + A^t = hh^t - I_n. \]
Consider the following matrices:

\[
A = \begin{bmatrix} 4 & 10 & 6 \\ -1 & 4 & 5 \\ 3 & 4 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

**Example**

A is a \(3\)-hypertournament matrix, where \(3 = 3 \cdot 1_3\).

B is a \(h\)-hypertournament matrix, where \(h = [0, 1, 1]^t\). (Note: \(\pm i\sqrt{2} \in \sigma(B)\))

C is a generalized tournament matrix.
Questions

- How do you rank players in a given tournament?
- When do you obtain the maximal spectral radius?
- Do tournament matrices have nonzero purely imaginary eigenvalues?
Notations

\[ \mathcal{M}_n \quad n \times n \text{ complex matrices.} \]

\[ \mathcal{M}_n(\mathbb{R}) \quad n \times n \text{ real matrices.} \]

\[ I_n \quad \text{the } n \times n \text{ identity matrix.} \]

\[ J_n \quad \text{the } n \times n \text{ all-ones matrix.} \]

\[ O_{k,m} \quad \text{the } n \times m \text{ zero matrix.} \]

\[ 1_n \quad \text{the } n \times 1 \text{ all-ones vector.} \]

\[ e_k \quad \text{the } k\text{-th standard basis vector for } \mathbb{R}^n \]
Notations

\[ \sigma(A) \quad \text{multi-set of eigenvalues of } A \in \mathcal{M}_n \]

\[ p_A(t) \quad \text{characteristic polynomial of } A \in \mathcal{M}_n \]

\[ \text{tr}(A) \quad \text{trace of } A \in \mathcal{M}_n \]

\[ \det(A) \quad \text{determinant of } A \in \mathcal{M}_n \]

\[ \rho(A) = \max_{\lambda \in \sigma(A)} |\lambda| \quad \text{spectral-radius of } A \in \mathcal{M}_n \]
**Notations**

- $A^t$: transpose of $A \in \mathcal{M}_n$
- $A^*$: conjugate transpose of $A \in \mathcal{M}_n$
- $A^{-1}$: inverse of $A \in \mathcal{M}_n$
- $\|x\|_2 = \sqrt{x^*x}$: Euclidean norm for $x \in \mathbb{C}^n$
Nonnegative Matrix

Definition

A matrix $A = [a_{ij}] \in M_n$ is said to be nonnegative if $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$, and it is denoted as $A \succeq 0$. 
Definition

Let \( x \) and \( y \) denote the right and left eigenvectors, respectively, of \( A \in \mathcal{M}_n(\mathbb{R}), A \geq 0 \), corresponding to \( \rho(A) \) which is simple.

When \( x \) and \( y \) are normalized so that

\[
1_n^t x = 1_n^t y = 1
\]

then \( x \) and \( y \) are unique and they are respectively referred to as the (right) \textbf{Perron vector} and the \textbf{left Perron vector} of \( A \).
Theorem (Perron-Frobenius)

Let $A \in \mathbb{T}_n$ be irreducible and nonnegative, then

i) $\rho := \rho(A) > 0$

ii) $\rho \in \sigma(A)$

iii) $\exists \ x > 0 \ s.t. \ Ax = \rho x$

iv) $\rho$ is simple
Bounding $\rho(A)$

**Theorem**

Let $A = [a_{jk}] \in \mathcal{M}_n(\mathbb{R})$ be nonnegative. Then, for any positive vector $x \in \mathbb{R}^n$ we have

$$\min_{1 \leq k \leq n} \frac{1}{x_k} \sum_{j=1}^{n} a_{kj} x_j \leq \rho(A) \leq \max_{1 \leq k \leq n} \frac{1}{x_k} \sum_{j=1}^{n} a_{kj}$$

and

$$\min_{1 \leq j \leq n} x_j \frac{1}{x_k} \sum_{k=1}^{n} a_{kj} x_k \leq \rho(A) \leq \max_{1 \leq j \leq n} x_j \sum_{k=1}^{n} \frac{a_{kj}}{x_k}.$$
Tournament Matrices: Example

Example

\[ A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \]

1 2 3

1 2 3 4

Matrix Analysis
All things in a linear algebraic manner

James Burk
Eigenspaces of Tournament Matrices
In a round-robin tournament $n$-players play against the other $(n - 1)$-players.

If player $i$ wins over player $j$ then $a_{ij} = 1$; furthermore, because $j$ losses under player $i$ then $a_{ji} = 0$.

The $i^{th}$ column sum is the number of losses for player $i$; whereas, the $i^{th}$ row sum is the number of wins for player $i$. 
Definition

A vector \( s \in \mathbb{R} \) is said to be the **score vector** for a given tournament matrix \( T \in T_n \) provided that \( s = T1 \), where \( 1 \) is the \( n \times 1 \) all-ones vector.
Score Vector

Example

\[ T1 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}, \quad 1 = \begin{bmatrix}
1 \\
1 \\
2 \\
2
\end{bmatrix} \]

\[ 1^t T = 1^t \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
2 \\
2 \\
1 \\
1
\end{bmatrix} \]
Who is the best player?

How can we rank the players?
Ranking Schemes

Score Ranking

1. Players are ranked according to the score vector.
2. Strengths of a player are represented by number of wins in the score vector.

Kendall-Wei Ranking

1. Players are ranked according to the (right) Perron vector.
2. Strengths of a player are represented by the (right) Perron vector.

Ramanajucharyula Ranking

1. Players are ranked according the right and left Perron vectors.
2. Strengths and Weaknesses are represented by the right and left Perron vectors, respectively.
Score vector leading to the Kendall-Wei scheme

Let $T$ be a given tournament matrix, and consider its score vector

$$s = T1.$$ 

Notice: the $i^{th}$ entry in $Ts$ represents the sum of the strengths of the players that player $i$ defeats.

$$r_2 = Ts = T(T1) = T^21.$$ 

Repeat this process for $T^2, T^3, \ldots$ in order to obtain the following sequence

$$r_1 = s, \quad r_2 = Ts, \quad r_3 = Tr_2 = T^2s, \ldots$$

$$\left\{r_k = T^{k-1}s\right\}_{k=1}^{\infty}$$

Power Method!
Ramanajucharyula ranking

Same idea as behind the Kendall-Wei scheme, but it now incorporates the weakness of the players.

Examines the ratio, strength-to-weakness:

$$\frac{x_k}{y_k},$$

where $Tx = \rho x$, $y^t T = \rho y^t$, and $\rho := \rho(A)$ for a given tournament matrix $T$. 
Theorem

Let $T$ and $\hat{T}$ be two $n \times n$ tournament matrices with Perron vectors $x$, $\hat{x}$, respectively. Let also $y$, $\hat{y}$ be left Perron vectors of $T$, $\hat{T}$, respectively. Then the following are equivalent.

(a) $\rho(T) \leq \rho(\hat{T})$  
(b) $\|x\|_2 \geq \|\hat{x}\|_2$  
(c) $\|y\|_2 \geq \|\hat{y}\|_2$

Furthermore, either in all of the above statements the inequalities are strict, or they all hold as equalities.
Proof

Let $T \in \mathbb{T}_n$ be a tournament matrix, and $x, y \in \mathbb{R}^n$ be right and left Perron vectors (respectfully).

\[ T + T^t = J_n - I_n \]

\[ x^t (T + T^t) x = x^t (J_n - I_n) x = x^t 1^t 1 - x^t x \]

\[ (2\rho)x^t x = 1 - x^t x \]

\[ \rho = \frac{1}{2\|x\|^2} - \frac{1}{2} \]

Notice that $\rho$ and $\|x\|_2^2$ are \textit{indirectly proportional}; therefore, $\rho$ increases $\Leftrightarrow$ $\|x\|_2^2$ decreases, thereby showing the equivalence of (a) and (b).

In parallel, same analysis can be used for $y$ instead of $x$, thereby, showing the equivalence of (a) and (c).
Regular and Almost Regular

**Definition**

Let $T \in \mathbb{T}_n$ be a given tournament matrix.

If $n$ is odd and all of the row sums of $T$ are equal $(n - 1)/2$, then $T$ is called **regular**.

If $n$ is even, and half of the row sums of $T$ equal $n/2$ and the rest equal $(n - 2)/2$, then $T$ is called **almost regular**.
Let $A \in T_{2m+1}$ be regular and let $B \in T_{2(m+1)}$ be almost regular, where $m > 1$. Then $A$ and $B$ share the following properties.

1. $A$ and $B$ are nonsingular
2. $A$ and $B$ are irreducible
3. $A$ and $B$ are primitive
Almost Regular: First Kind

Definition

Given $T \in \mathbb{T}_m$, and construct the tournament matrix $A \in \mathbb{T}_{2m}$ as

$$A = \begin{bmatrix} T & T^t \\ T^t + I_m & T \end{bmatrix}.$$ 

The matrix $A$ will be referred to as an almost regular of the first kind, which is generated by $T$. 
The Brualdi-Li Matrix

Definition

The Brualdi-Li tournament matrix is defined as

\[ B_{2m} = \begin{bmatrix} L & L^t \\ L^t + I_m & L \end{bmatrix}, \]

where \( L \) denotes the \( m \times m \) strictly lower triangular tournament matrix, i.e. all of the entries below the main diagonal are equal to one.
Brualdi-Li Examples

Example

\[ \mathcal{B}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathcal{B}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \]

\[ 1.3660 < \rho(\mathcal{B}_4) = 1.3953 < 1.4063 \]

\[ 2.4142 < \rho(\mathcal{B}_6) = 2.4340 < 2.4375 \]
Regular: Perron Vectors

Recall that regular tournaments are normal.

**Theorem**

Let $A \in \mathcal{M}_n$ be a normal matrix. Then $x \in \mathbb{C}^n$ is a right eigenvector of $A$ corresponding to the eigenvalue of $\lambda$ of $A$ if and only if $x$ is a left eigenvector of $A$ corresponding to $\lambda$. 
Almost Regular: Perron Vectors

Is there an analog for the Perron vectors for a given almost regular?
Brualdi-Li: Perron Vectors

Theorem

Let $x$ and $y$ respectively be the right and left Perron vector for $B_{2m}$. Then $y = C_{2m}x$.

Why? Because $C_{2m}^t B_{2m}^t C_{2m} = B_{2m}$, where

$$C_{2m} = \begin{bmatrix} R_m & O_m \\ O_m & R_m \end{bmatrix}$$

and $R_m$ is the $m \times m$ reverse identity matrix.
The Structure of the Brualdi-Li

What makes the Brualdi-Li interesting?

1. Toeplitz
2. Persymmetric
3. Almost Circulant
4. We have explicitly calculated the inverse.
Brualdi-Li Inverse

Let $I_k$ and $J_k$ denote the $k \times k$ identity matrix and all-ones matrix, respectively. Let also $m \geq 2$ and consider the $m \times m$ anti-diagonal matrix $P = [p_{ij}]$ defined by $p_{ij} = \begin{cases} 1 & \text{if } i + j = m + 1 \\ 0 & \text{otherwise}, \end{cases}$

Define the matrices

$$A = \begin{bmatrix} J_{m-1} - (m-1)I_{m-1} & 1_{m-1} \\ 0^t_{m-1} & 1 - m \end{bmatrix}$$

$$C = \begin{bmatrix} 0_{m-1} & J_{m-1} - (m-1)I_{m-1} \\ m-1 & 0^t_{m-1} \end{bmatrix}.$$

Then,

$$B_{2m}^{-1} = \frac{1}{m-1} \begin{bmatrix} A & C \\ J_{m} - (m-1)I_{m} & PA^tP \end{bmatrix}. $$
**Proof Outline**

A matrix is given, and then it is shown that it works!

How was this pattern derived?

Seeing a pattern after computing lots of examples.
The Brualdi-Li Conjecture

Conjecture

If $T \in \mathcal{M}_{2m}$ is a tournament matrix, then $\rho(T) \leq \rho(B_{2m})$. 
The Brualdi-Li Conjecture Affirmation

Theorem

If $T \in \mathcal{M}_{2m}$ is a tournament matrix, then $\rho(T) \leq \rho(B_{2m})$.

Perron Eigenspace of the Brualdi-Li
Future Research

Eigenvalues for First-Kind Almost Regular

Theorem

Let \( v = [v_j] \in \mathbb{R}^m \) and \( w = [w_j] \in \mathbb{R}^m \) so that \( x = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^{2m} \) is an eigenvector of the tournament matrix \( T = \begin{bmatrix} T \\ T^t + I_m \\ T \end{bmatrix} \) corresponding to \( \lambda \). Then,

\[
\lambda = \frac{1^{t}_{2m}x - w_k}{v_k + w_k}.
\]
Proof

First notice that $Tx = \lambda x$ yields

\[
\begin{bmatrix}
\frac{\lambda v}{\lambda w} \\
\lambda w
\end{bmatrix} = \begin{bmatrix}
T & T^t \\
T^t + I_m & T
\end{bmatrix} \begin{bmatrix}
v \\
w
\end{bmatrix}
\]

\[
= \begin{bmatrix}
Tv + T^tw \\
T^tv + v + Tw
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(J_m - I_m - T^t)v + T^tw \\
T^tv + v + Tw
\end{bmatrix}.
\]

Equating these blocks in order to obtain the following results.
**Proof**

\[ J_m v + T^t w + Tw = \lambda(w + v) \]
\[ \implies J_m v + (J_m - I_m)w = \lambda(w + v) \]
\[ \implies 1^t(1^t(v + w)) - w = \lambda(w + v) \]
\[ \implies 1^t1(1^t(w + v)) - 1^tw = \lambda(1^t(w + v)) \]
\[ \implies m - \frac{1^tw}{1^t(w + v)} = \lambda. \]
Proof

Furthermore, the equation

$$1 \left(1^t(v + w)\right) - w = \lambda(w + v)$$

yields

$$\lambda(w_k + v_k) = 1 \left(1^t(v + w)\right) - w_k$$

$$\implies \lambda = \frac{1 \left(1^t(v + w)\right) - w_k}{(w_k + v_k)}.$$

And since $1^t_{2m} = 1^t_m(v + w)$ then we are done.
Perron Eigenvectors

Notation

Let \( v = [v_j] \in \mathbb{R}^m \) and \( w = [w_j] \in \mathbb{R}^m \) so that \( x = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^{2m} \) is the Perron vector of \( \mathcal{B}_{2m} \).

Furthermore, \( y \in \mathbb{R}^{2m} \) shall denote the left-Perron vector.
Eigenvector: Ordering

**Theorem**

\[ v_m < v_{m-1} < \ldots < v_1 < w_1 < w_2 < \ldots < w_m. \]

Kendall-Wei Interpretation:

The player corresponding to \( w_m \), i.e. player \( 2m \) is the strongest, followed (in order) by players \( 2m - 1, 2m - 2, \ldots, m + 1, 1, 2, \ldots, m \).
Proof Outline

Since $B_{2m}x = \rho x$, we have that for $k = 1, 2, \ldots, m - 1$,

$$\rho v_k = \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j$$

from which it follows that for all $k = 1, 2, \ldots, m - 1$,

$$\rho v_k - \rho v_{k+1} = \left(\sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j\right) - \left(\sum_{j=1}^{k} v_j + \sum_{j=k+2}^{m} w_j\right)$$

$$= w_{k+1} - v_k.$$

Applying a proposition in the paper, yields for $k = 1, 2, \ldots, m$,

$$v_{k+1} < v_k.$$
Because \( w_k = \frac{1 - \rho v_k}{\rho + 1} \) for each \( k = 1, \ldots, m \) and since \( 0 \leq v_{k+1} < v_k \) for \( k = 1, 2, \ldots, m - 1 \), then

\[ w_k < w_{k+1} \quad \text{for} \quad k = 1, 2, \ldots, m - 1. \]

Then we apply a result from Kirkland that states

\[ v_j < w_k \quad \text{for} \quad j, k = 1, \ldots, m \]

in order to complete the proof.
Eigenvector: Explicit

**Theorem**

\[
\begin{align*}
v_1 &= \frac{(m - 1)(\rho + 1) - \rho^2}{\rho^2} \\
v_m &= \frac{\rho + 1 - m}{\rho + 1} \\
v_{k+1} &= \frac{v_k(\rho + 1)^2 - 1}{\rho^2}
\end{align*}
\]

\[
\begin{align*}
w_1 &= \frac{\rho + 1 - m}{\rho} \\
w_m &= \frac{1 + m\rho - \rho^2}{(\rho + 1)^2} \\
w_{k+1} &= \frac{w_k(\rho + 1)^2 - 1}{\rho^2}
\end{align*}
\]

for \( k = 1, 2, \ldots, m - 1 \).

Recall: \( Bx = \rho x \) and \( y^t B = \rho y^t \).
Proof Outline

As a proof, we make use of the fact that

\[ \rho v_k = \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j. \]

Then we examine the ratios

\[ \frac{w_\ell - v_\ell}{w_j - v_j} = \left( \frac{\rho + 1}{\rho} \right)^{2(\ell-j)}. \]

Then we apply use a fact from Kirkland [?] that states

\[ 2\rho^2 - 2\rho(m - 1) - (m - 1) = \left( \left( \frac{\rho + 1}{\rho} \right)^{2m} + 1 \right)^{-1}. \]

And, then lots of algebra manipulations.
Interlaced Relationship: Strength to Weakness

Theorem

\[
\frac{x_m}{y_m} < \frac{x_1}{y_1} < \frac{x_{m-1}}{y_{m-1}} < \frac{x_2}{y_2} < \cdots < \frac{x_{\lceil m/2 \rceil}}{y_{\lceil m/2 \rceil}} < 1 < \\
\frac{x_{2m-\lfloor m/2 \rfloor} + 1}{y_{2m-\lfloor m/2 \rfloor} + 1} < \cdots < \frac{x_{m+2}}{y_{m+2}} < \frac{x_{2m}}{y_{2m}} < \frac{x_{m+1}}{y_{m+1}}.
\]
Proof Outline

1. Define the function $f(k) = \frac{y_k}{x_k}$, for $k = 1, \ldots, m$.

2. Show $f(m) > f(m - j)$, for $j = 1, \ldots, m - 1$.

3. Show $f(j) > f(m - j)$, for $j = 1, \ldots, \lceil m/2 \rceil$.

4. Show $f(m - j) > f(j + 1)$, for $j = 1, \ldots, \lceil m/2 \rceil$. 
Interlaced Relationship

Example

After computing the right Perron, \( x \), and the left Perron, \( y \), vectors for \( B_{12} \) we have the following interlaced relationship

\[
\begin{align*}
\frac{x_6}{y_6} &= 0.8454 < \frac{x_1}{y_1} = 0.8738 < \frac{x_5}{y_5} = 0.8761 < \frac{x_2}{y_2} = 0.8910 < \\
&< \frac{x_4}{y_4} = 0.8927 < \frac{x_3}{y_3} = 0.8973 \\
&< 1 < \\
\frac{x_{10}}{y_{10}} &= 1.1144 < \frac{x_9}{y_9} = 1.1202 < \frac{x_{11}}{y_{11}} = 1.1224 < \frac{x_8}{y_8} = 1.1414 < \\
&< \frac{x_{12}}{y_{12}} = 1.1444 < \frac{x_7}{y_7} = 1.1829.
\end{align*}
\]
Strength of Players: Increasing to Decreasing

Example

Score Vector
Players 1 to 6 are equal strength, and Players 7 to 12 are equal strength. Players 7 to 12 are stronger than players 1 to 6.

Kendall-Wei Ranking

12, 11, 10, 9, 8, 7, 1, 2, 3, 4, 5, 6.

Ramanajucharyula Ranking

7, 12, 8, 11, 9, 10, 3, 4, 2, 5, 1, 6.
Theorem

Let $v = [v_j] \in \mathbb{R}^m$ and $w = [w_j] \in \mathbb{R}^m$ ($m \geq 2$) so that $x = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^{2m}$ is the Perron vector of $B_{2m}$ and let $\rho = \rho(B_{2m})$; furthermore, let $y$ denote the left Perron vector of $B_{2m}$. Then

$$y^t x = 2 \left( m + \frac{1}{2\rho + 1} - \frac{m^2(m+1)}{\rho + 1} + \frac{m^2(m-1)}{\rho} \right).$$
Separable Differential Equation

\[ \frac{\partial \rho}{\partial b_{ij}} = \frac{y_i x_j}{y^t x}. \]

By choosing \( i = m \) and \( j = m + 1 \), leads to

\[ \Rightarrow \frac{\partial \rho}{\partial b_{ij}} = \frac{v_m w_1}{y^t x}. \]

Replacing the \((m, m + 1)\) entry in \( B_{2m} \) by the variable \( t \), then when \( t = 0 \) we have

\[ \frac{d \rho}{dt} \bigg|_{t=0} = \frac{(\rho - m + 1)^2}{\rho(\rho + 1)} \left( 2(m + \frac{1}{2\rho + 1} - \frac{m^2(m + 1)}{\rho + 1} + \frac{m^2(m - 1)}{\rho}) \right)^{-1} \]
Solving this separable differential equation yields:

\[
2m\rho - \frac{\ln (2\rho + 1)}{(2m - 1)^2} - \frac{2(m - 1) (4m^2 - 2m + 1) \ln (\rho - m + 1)}{(2m - 1)^2} - \frac{2m(m - 1)}{(2m - 1)(\rho - m + 1)} = C
\]

Can we independently evaluate \( C \)?
Choose a vector $z \in \mathbb{R}^{2m}$ with a component in the direction of the Perron vector $x$ for $B_{2m}$.

$$\lim_{k \to \infty} \frac{B^k z}{\mathbf{1}_{2m}^t B^k x} = x$$

$$\lim_{k \to \infty} \frac{B^{k+1} z}{\mathbf{1}_{2m}^t B^k x} = Bx = \rho x$$

$$\lim_{k \to \infty} \frac{\mathbf{e}_{m+1} B^{k+1} z}{\mathbf{1}_{2m}^t B^k x} = \rho \mathbf{e}_{m+1} x$$

$$= \rho \mathbf{w}_1 = \rho + 1 - m$$
Power Method

\[ \rho = (m - 1) + \lim_{k \to \infty} \frac{e_{m+1} B^{k+1} z}{1^t_{2m} B^k x} \]

What is a good choice for \( z \) that will allow us to “easily” compute this limit?
**Theorem**

\[
\begin{align*}
\mathbf{v}_1 &= \left( \frac{(m-1)(\rho + 1) - \rho^2 + \gamma(2m(\rho + 1) + (\rho - 1))}{\rho^2} \right) (\mathbf{I}_m^t(\mathbf{v} + \mathbf{w})) \\
\mathbf{v}_m &= \left( \frac{\rho + 1 - m - \gamma(2m - 1)}{\rho + 1} \right) (\mathbf{I}_m^t(\mathbf{v} + \mathbf{w})) \\
\mathbf{v}_{k+1} &= \left( \frac{(\rho + 1)^2\mathbf{v}_k - (2\gamma + 1)(\mathbf{I}_m^t(\mathbf{v} + \mathbf{w}))}{\rho^2} \right)
\end{align*}
\]

for \(k = 1, 2, \ldots, m - 1\).

Recall: \(B\mathbf{x} = \rho\mathbf{x}\) and \(\mathbf{y}^tB = \rho\mathbf{y}\)

where \(\rho := \rho(B)\) and \(B = B_{2m} + \gamma J_{2m}\).
Eigenvector: Explicit

Theorem

\[ w_1 = \left( \rho + 1 - m - \gamma(2m - 1) \right) \frac{1}{\rho} (1^t_m(v + w)) \]
\[ w_m = \left( m\rho + 1 - \rho^2 + \gamma(\rho + 2m\rho + 2) \right) \frac{1}{(\rho + 1)^2} (1^t_m(v + w)) \]
\[ w_{k+1} = \frac{w_k(\rho + 1)^2 - (1 + 2\gamma)(1^t_m(v + w))}{\rho^2} \]

for \( k = 1, 2, \ldots, m - 1 \).

Recall: \( Bx = \rho x \) and \( y^t B = \rho y^t \)
where \( \rho := \rho(B) \) and \( B = B_{2m} + \gamma J_{2m} \).
Closed Form: $\rho (B_{2m} + \gamma J_{2m})$

**Theorem**

Let $C = B_{2m} + \gamma J_{2m}$ where $\gamma \in \mathbb{C}$, and $(\lambda, x)$ be an eigenpair of $C$; furthermore, define $v, w \in \mathbb{C}^m$ such that $x = \begin{bmatrix} v \\ w \end{bmatrix}$. Then,

$$\left( \left( \frac{\rho + 1}{\rho} \right)^{2m} + 1 \right)^{-1} = \frac{(2\rho + 1)^2}{2(2\gamma + 1)} - m(2\rho + 1) + 1/2,$$

where $\rho := \rho (B_{2m} + \gamma J_{2m})$. 
Definition

If $\mathcal{T} = \mathcal{T}(A, B)$ is a tally tournament where $B = \mathcal{B}_m$ and $A \in \mathcal{T}_n$ is a regular tournament, then we call $\mathcal{T}$ the Brualdi-Li tally (tournament) of order $m$. 
Conjecture

Let $nm \in \mathbb{N}$ be a nontrivial composite even integer and $\eta$ be the smallest prime factor of $nm$.

Let $T \in \mathcal{T}_{nm}$ be any tournament, which is not permutationally similar to $\mathcal{B}_{nm}$.

If there exists a Brualdi-Li tally, $\mathcal{T}$, of order $(nm)/\eta$ then

$$\rho(T) \leq \rho(\mathcal{T}) < \rho(\mathcal{B}_{nm}).$$
What is the connection between $\rho(B_{2m})$ and $\rho(B_{2m} + \gamma J_{2m})$?

Recall

$$\left(\left(\frac{\rho + 1}{\rho}\right)^{2m} + 1\right)^{-1} = \frac{(2\rho + 1)^2}{2(2\gamma + 1)} - m(2\rho + 1) + 1/2,$$

where $\rho := \rho(B_{2m} + \gamma J_{2m})$. 
A matrix $A \in \mathcal{M}_n$ is said to be *traceless* if $\text{tr}(A) = 0$. 
Let \( A \in \mathcal{M}_n \) be a traceless matrix such that \( \rho(A) \in \sigma(A) \). The real and imaginary parts of the eigenvalues of \( A \) are denoted by

\[
\alpha_k = \Re(\lambda_k), \quad \text{and} \quad \beta_k = \Im(\lambda_k), \quad \text{where} \quad \lambda_k \in \sigma(A).
\]

We will order the real parts as

\[
\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n.
\]
Traceless Notation

Notation

The variables $\pi, \tau, \ell$ will be reserved to denote the following quantities.

$\pi$ denotes the number of eigenvalues such that $\text{Re}(\lambda) = \rho(A)$.

$\tau$ denotes the number of eigenvalues such that $0 < \text{Re}(\lambda) \neq \rho(A)$.

$\ell$ denotes the number of eigenvalues such that $\text{Re}(\lambda) = 0$. 
Traceless Notation

Notation

\[ \alpha_1 \leq \cdots \leq \alpha_{n-\tau-\ell-\pi} < \alpha_{n-\tau-\ell-\pi+1} = \cdots = \alpha_{n-\tau-\pi} = 0 \]

\[ \leq \alpha_{n-\tau-\pi+1} \leq \cdots \leq \alpha_{n-\pi} < \alpha_{n-\pi+1} = \cdots = \alpha_n. \]

In addition, we will also define

\[ \alpha_M = \max_{1 \leq k \leq n-\tau-\ell-\pi} |\alpha_k|. \]
Let $A \in \mathcal{M}_n(\mathbb{R})$ be a real traceless matrix.

If $\alpha_n \in \sigma(A)$, $2 < n$, and $\kappa(n - 2)\alpha_M \leq \alpha_n$, where $\kappa \in \mathbb{R}$.

If $1 \leq \pi \kappa$ then the spectrum of $A$ has the following properties:

a) The spectrum of $A$ does not contain a nonzero purely imaginary eigenvalue.

b) If $\lambda \in \sigma(A)$ then either $\text{Re}(\lambda) < 0$ or $\text{Re}(\lambda) = \alpha_n$.

c) Either $\pi = 1$ or $\pi = 2$.

Furthermore, if $0 \in \sigma(A)$ then $\text{am}_A(0) = 1$. 
Necessary Condition

Theorem

If there exists a tournament matrix, $A \in T_n$, such that it contains a nonzero purely imaginary eigenvalue, then we can infer the following about $A$.

(a) $A$ must be an irreducible matrix such that $\rho(A) < \frac{n - 2}{2}$.
(b) $A$ cannot contain a regular submatrix of size $(n - 1)$. 
Thank You.